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Also solved by H. C. FEEMSTER, J. A. CAPARO, CLIFFORD N. MILLS, and HORACE OLSON. Erroneously solved by HERBERT N. CARLETON.

306. Proposed by EMMA M. GIBSON, Drury College.

A sphere is composed of a solid homogeneous hemisphere and a very thin hemispherical shell of equal mass. What is the greatest inclination of a rough plane on which the sphere can just rest in equilibrium?

SOLUTION BY PAUL CAPRON, U. S. Naval Academy.

Let the radius of the sphere be a, the center at O, the center of gravity at G; then

$$OG = \frac{1}{2}\left(\frac{a}{2} - \frac{3a}{8}\right) = \frac{a}{16};$$

however the sphere is placed, G is on a circumference of radius $a/16$, centered at O. If the sphere is in equilibrium, G is vertically over the contact, C, of the sphere with the inclined plane on which it rests; the inclination of the plane is the angle COG, which has its greatest value when CG is tangent to the little circle; this value is $\sin^{-1} 1/16$. Hence, the greatest inclination of the plane is $\sin^{-1} 1/16$ or $\tan^{-1} \mu$ (μ being the coefficient of friction) whichever happens to be the less.

Also solved by J. A. CAPARO. Erroneously solved by HERBERT N. CARLETON.

NUMBER THEORY.

226. Proposed by ELBERT H. CLARKE, Purdue University.

If $0!$ is taken equal to 1, and if k is any positive integer greater than or equal to 2, show that

$$\sum_{n=0}^{\infty} \frac{n!}{(k+n)!} = \frac{1}{k-1} \cdot \frac{1}{(k-1)!}.$$

I. SOLUTION BY S. A. JOFFE, New York City.

Since

$$\frac{n!}{(k+n)!} = \frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 2 \cdot 3 \cdots n(n+1)(n+2) \cdots (n+k)} = \frac{1}{(n+1)(n+2) \cdots (n+k)},$$

we see that

$$\sum_{n=0}^{\infty} \frac{n!}{(k+n)!} = \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2) \cdots (n+k)}. \tag{1}$$

Now, the series in the second member, each term of which is the reciprocal of a factorial expression, may be easily summed by an elementary theorem in finite differences or by the following equivalent simple method.

Noticing that

$$\frac{1}{(n+1)(n+2) \cdots (n+k-1)} - \frac{1}{(n+2)(n+3) \cdots (n+k)} = \frac{(n+k)-(n+1)}{(n+1)(n+2) \cdots (n+k)} = \frac{k-1}{(n+1)(n+2) \cdots (n+k)},$$

we find that

$$\frac{1}{(n+1)(n+2) \cdots (n+k)} = \frac{1}{k-1}\left[\frac{1}{(n+1)(n+2) \cdots (n+k-1)} - \frac{1}{(n+2)(n+3) \cdots (n+k)}\right].$$

Making in the last equation n equal successively to 0, 1, 2, $\cdots$, we get

$$\frac{1}{1\cdot 2\cdots k}=\frac{1}{k-1}\left[\frac{1}{1\cdot 2\cdots (k-1)}-\frac{1}{2\cdot 3\cdots k}\right],$$

$$\frac{1}{2\cdot 3\cdots (k+1)}=\frac{1}{k-1}\left[\frac{1}{2\cdot 3\cdots k}-\frac{1}{3\cdot 4\cdots (k+1)}\right],$$

$$\frac{1}{3\cdot 4\cdots (k+2)}=\frac{1}{k-1}\left[\frac{1}{3\cdot 4\cdots (k+1)}-\frac{1}{4\cdot 5\cdots (k+2)}\right],$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

the first of which is true only if $k-1$ is at least equal to 1, that is $k \geqq 2$.

If we add these equations and observe that the second term in the brackets in each equation cancels the first term in the following equation, so that in the second member there will remain only the first term of the first equation, we shall obtain

$$\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)\cdots(n+k)}=\frac{1}{k-1}\cdot\frac{1}{1\cdot 2\cdots (k-1)}=\frac{1}{k-1}\cdot\frac{1}{(k-1)!},$$

and in view of equation (1) this becomes

$$\sum_{n=0}^{\infty}\frac{n!}{(k+n)!}=\frac{1}{k-1}\cdot\frac{1}{(k-1)!},$$

as required.

II. Solution by L. C. Mathewson, Dartmouth College.

The statement to be proved may be put into the following form:

$$\frac{1}{k-1}\cdot\frac{1}{(k-1)!}=\frac{0!}{k!}+\frac{1!}{(k+1)!}+\frac{2!}{(k+2)!}+\cdots+\frac{n!}{(k+n)!}+\cdots. \tag{1}$$

In the first place, if k is any finite positive integer, the series in the second member is a convergent series as may be easily shown by Gauss's test;[1] for, the ratio of the $(n+1)$th term to the $(n+2)$th is

$$\frac{\alpha_{n+1}}{\alpha_{n+2}}=\frac{n+(k+1)}{n+1},$$

and since by hypothesis $k>1$, $(k+1)-1>1$, and the series is convergent, and absolutely so.

Assuming that any positive rational integer can be attained by successive additions of unity, a proof of the given formula will be made by use of mathematical induction.

If $k=2$, there results

$$\begin{aligned}1&=\frac{0!}{2!}+\frac{1!}{3!}+\frac{2!}{4!}+\cdots+\frac{n!}{(n+2)!}+\cdots\\&=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(n+1)(n+2)}+\cdots\\&=(\tfrac{1}{1}-\tfrac{1}{2})+(\tfrac{1}{2}-\tfrac{1}{3})+(\tfrac{1}{3}-\tfrac{1}{4})+\cdots+\left(\frac{1}{n+1}-\frac{1}{n+2}\right)+\cdots\\&=1-\tfrac{1}{2}+\tfrac{1}{2}-\tfrac{1}{3}+\tfrac{1}{3}-\cdots-\frac{1}{n+1}+\frac{1}{n+1}-\frac{1}{n+2}-\cdots,\end{aligned} \tag{2}$$

since the commutative law holds for an absolutely convergent series.[2] From this the sum of the first $n+1$ terms of the series is $1-1/(n+2)$, which approaches the limit 1 as n becomes infinite. Hence, the formula is true for $k=2$.[3]

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[1] Pierpont, *Theory of Functions of Real Variables*, volume 2, article 97.

[2] Pierpont, *loc. cit.*, article 110, 2.

[3] Cf. Chrystal, *Algebra*, Part II, page 120, footnote.

Now assume the given statement true if $k = r$; *i. e.*, that

$$\frac{1}{r-1}\cdot\frac{1}{(r-1)!}=\frac{0!}{r!}+\frac{1!}{(r+1)!}+\frac{2!}{(r+2)!}+\cdots+\frac{n!}{(r+n)!}+\cdots. \tag{3}$$

Transposing the first term of the second member of (3),

$$\frac{1}{r-1}\cdot\frac{1}{(r-1)!}-\frac{0!}{r!}=\frac{1!}{(r+1)!}+\frac{2!}{(r+2)!}+\cdots+\frac{r!}{(r+n)!}+\cdots, \tag{4}$$

where the second member is an absolutely convergent series. But

$$\frac{1}{r-1}\cdot\frac{1}{(r-1)!}-\frac{0!}{r!}=\frac{\dfrac{r}{r-1}-1}{r!}=\frac{1}{r!(r-1)}.$$

Accordingly, (4) becomes

$$\frac{1}{r!(r-1)}=\frac{1!}{(r+1)!}+\frac{2!}{(r+2)!}+\frac{3!}{(r+3)!}+\cdots+\frac{n!}{(r+n)!}+\frac{(n+1)!}{(r+n+1)!}+\cdots. \tag{5}$$

Subtracting (5) from (3) gives, since the series are convergent,[1]

$$\begin{aligned}\frac{1}{r-1}\cdot\frac{1}{(r-1)!}-\frac{1}{r!(r-1)}&=\left(\frac{0!}{r!}-\frac{1!}{(r+1)!}\right)+\left(\frac{1!}{(r+1)!}-\frac{2!}{(r+2)!}\right)\\&\quad+\left(\frac{2!}{(r+2)!}-\frac{3!}{(r+3)!}\right)+\cdots+\left(\frac{n!}{(r+n)!}-\frac{(n+1)!}{(r+n+1)!}\right)+\cdots\\&=\frac{1!r}{(r+1)!}+\frac{1!(2+r)-2!}{(r+2)!}+\frac{2!(3+r)-3!}{(r+3)!}+\cdots+\frac{n!(n+1+r)-(n+1)!}{(r+n+1)!}+\cdots\\&=r\left\{\frac{0!}{(r+1)!}+\frac{1!}{(r+2)!}+\frac{2!}{(r+3)!}+\cdots+\frac{n!}{(r+n+1)!}+\cdots\right\}.\end{aligned} \tag{6}$$

Now the first member of (6) may be simplified:

$$\frac{1}{r-1}\cdot\frac{1}{(r-1)!}-\frac{1}{r!(r-1)}=\frac{r-1}{r!(r-1)}=\frac{1}{r!}.$$

Accordingly, dividing each member of (6) by r,

$$\frac{1}{r!r}=\frac{0!}{(r+1)!}+\frac{1!}{(r+2)!}+\frac{2!}{(r+3)!}+\cdots+\frac{n!}{(r+n+1)!}+\cdots.$$

But this is exactly what (1) becomes if $r + 1$ is substituted for k. Hence, if (1) is true for a particular integral value of k ($k > 1$), then it is true for k equal to the next greater integer. But (1) was shown true if $k = 2$, accordingly, it is true for $k = 3$; if for $k = 3$, then for $k = 4$; etc.

Also solved by HORACE OLSON, FRANK IRWIN, and ELIJAH SWIFT.

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[1] Pierpont, *loc. cit.*, article 111.